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Non-canonical subgroup restrictions of the unitary group U(nm)↓U(n)⊗U(m)

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Abstract. A reduction of the inner product space of the permutation group on an N-particle system has been used to develop a scheme for obtaining the states spanning a general irreducible representation of the unitary group U(nm) in terms of those spanning the product representations of U(n)⊗U(m). Since relatively straightforward methods are now available for determining the Clebsch–Gordan coefficients for the permutation groups, the present procedure is viable. As special cases of the general expression, algebraic expressions for the reduction of identity and alternating representations of U(nm) are obtained.

1. Introduction

The adaptation of N-particle basis states spanning an irreducible representation (IRREP) of the unitary group U(nm) to the bases spanning the product representations of the subgroup U(n)⊗U(m) and the similar problem for the special unitary group SU(nm) ⊃ SU(n)⊗SU(m) has an important role to play in both elementary particle physics and nuclear physics. Thus simple quark models (cf Lichtenberg 1970 and references therein) and the multiquark models (Jaffe 1977) make use of the restriction SU(6) ⊃ SU(3)⊗SU(2). Inclusion of charm leads to consideration of SU(8) ⊃ SU(4)⊗SU(2) (Greenberg 1978). In nuclear physics, the well known Wigner supermultiplet scheme (Wigner 1937) involves the study of SU(4) ⊃ SU(2)⊗SU(2).

In spite of the above, not many detailed studies of these restrictions are available in literature. Most of the earlier studies (Baird and Biedenharn 1963, Nagel and Moshinsky 1965a,b, Louck 1970) have primarily been concerned with extracting the symmetric content of U(nm) from the product representations of U(n)⊗U(m). In a more recent note Patterson and Harter (1976b) were able to generate both the identity and the alternating representations of U(nm) from those of the product representations of U(n)⊗U(m). Their approach involved the use of 'seminormal projection operators' which act as permutation operators on the single particle orbital labels providing the fundamental representation spaces of either U(n) or U(m). There are in the main two limitations in this approach even though it leads to a normalisable orthogonal basis. The first is that only the least troublesome IRREPS of U(nm) can be handled using this technique. Secondly the seminormal operators involve explicitly the orbital indices which implies the need to realise these operators for the generally large dimensional irreducible spaces of U(nm). The complexity of the problem tends to increase rapidly with increasing number of particles and also with increasing n and m. A more recent alternative (Strottman 1979) is an extremely elegant formulation but involves the realisation of quite a large number of coefficients of fractional parentage. This approach has been applied to the restriction $SU(6) \supset SU(3) \otimes SU(2)$ but the more general problem of arbitrary n and m has not been handled.

In the present paper we have exploited the dualism between the permutation and the unitary groups (cf Weyl 1956, Bohr and Mottelson 1969, Robinson 1961) to obtain a viable scheme for the study of unitary group restrictions. The part of the dualism that concerns us relates to the fact that the study of inner product reduction of the representations of $S_N \times S_N$ should lead to a systematic procedure for the product representations of $U(n) \otimes U(m)$ (Bohr and Mottelson 1969). Since reasonably efficient schemes are available (Schindler and Mirman 1977, Sahasrabudhe *et al* 1981) for obtaining the $S_N \times S_N$ reduction, a procedure based on the dualism should be viable. A recent study (Sarma and Sahasrabudhe 1980) of permutation adapted canonical tensor bases for the IRREPS of unitary groups helps considerably in separating out the $S_N \times S_N$ content of the required restriction so that the need to handle large dimensional IRREPS of $U(nm)$ is not there.

The procedure has been outlined in § 2 and illustrated using examples. Both the identity and alternating representations of $U(nm)$ have been displayed in a closed algebraic form valid for any n, m and N. A brief discussion has been presented in § 3.

2. Non-canonical basis for U(*nm*) adapted to U(*n*)⊗U(*m*)

First, we briefly review the procedure for generating the canonical tensor bases spanning the IRREPS of the unitary groups in the context of $U(nm)$ (cf Sarma and Sahasrabudhe 1980 for details). Let $\{\chi_{ij}(\alpha)|ij = 11, 12, \ldots nm; \alpha = 1, \ldots N\}$ define an ordered orthonormal single particle basis spanning the fundamental representation space V_{nm} of $U(nm)$. The ordering of the basis is defined in the sense that χ_{ij} precedes $\chi_{i'j'}$ for any j, j' if $i < i'$ and for $j < j'$ if $i = i'$. Using these basis orbitals the space of Nth rank tensors, $V_{nm} \otimes^N$, is generated in the form of ordered products as

$$V_{nm} \otimes^N : \left\{ N_{(ij)} \equiv |(N_{11} \ldots N_{nm})\rangle = |\chi_{11}\rangle^{N_{11}} \ldots |\chi_{nm}\rangle^{N_{nm}}; \sum_{ij=11}^{nm} N_{ij} = N \right\}. \tag{1}$$

This tensor space is reducible into irreducible subspaces of $U(nm)$. The reduction leading to canonical basis states spanning the IRREP $[\lambda]$ of $U(nm)$ can be effected using the Wigner operators of S_N which can be defined in a normalised form (Kaplan 1975) as

$$\omega^{\lambda}_{rs} = \left(\frac{f_\lambda}{N!}\right)^{1/2} \sum_{P \in S_N} [P]^{\lambda}_{rs} P \tag{2}$$

where $[\lambda]$ is the corresponding IRREP of S_N, $[P]^{\lambda}_{rs}$ is the rsth element of the Young orthogonal representation matrix (Robinson 1961) for the permutation P and f_λ is the dimensionality of the IRREP of S_N. Applying the above operator to the monomial $|(N_{11} \ldots N_{nm})\rangle$ of equation (1) we observe that linear dependencies arise. This follows since any permutation Q which leaves the monomial invariant transforms ω^{λ}_{rs} to a linear combination of $\omega^{\lambda}_{rs'}$. A consistent choice of operators in this ambiguous situation is to define a symmetrised normalised combination,

$$\omega^{\lambda}_{r(p)} = \sum_{s \in S(p)} a^{\lambda}_{s(p)} \omega^{\lambda}_{rs} \tag{3}$$

such that

$$\omega^{\lambda}_{r(p)}Q=\omega^{\lambda}_{r(p)} \tag{4}$$

for any permutation Q belonging to the subgroup $S_{N_{11}}\otimes S_{N_{12}}\otimes\ldots\otimes S_{N_{nm}}$ of the permutation group S_N. The suffix (p) has been used in equation (3) to represent a standard Weyl tableau of the IRREP $[\lambda]$ of $U(nm)$ containing as entries the indices of orbitals occurring in the monomial $|(N_{11}N_{12}\ldots N_{nm})\rangle$ of equation (1). The summation on the right of equation (3) is over a set $S(p)$ of standard Young tableaux (SYT) s such that the replacement of the entries $1,2,\ldots,N$ in s by the orbital indices $11,12,\ldots,nm$ as they occur in $|(N_{11}N_{12}\ldots N_{nm})\rangle$ yields the standard Weyl tableau (p) (cf figure 6 of Patterson and Harter 1976a and p 119 of Bohr and Mottelson 1969). Representing the monomial $|(N_{11}\ldots N_{nm})\rangle$ more compactly as $|N_{(ij)}\rangle$ we observe that the Weyl tableau index (p) can be replaced by a combination index $(s:N_{(ij)})$ since such a combination corresponds to a unique (p) which, in turn, fixes the complete set $S(p)$ of SYT. The symmetrisation coefficients $a^{\lambda}_{s(p)}$ of equation (3) can readily be determined using elementary transpositions of $S_{N_{11}}\otimes S_{N_{12}}\otimes\ldots\otimes S_{N_{nm}}$ as outlined in an earlier note for $U(n)$ (cf the discussion leading to equation (19) of Sarma and Sahasrabudhe (1980)). As an illustration of the procedure, consider

$$\omega^{[3,1]}_{r\binom{122}{2}}$$

for the IRREP [3, 1] of U(2). We observe that the set $S(p)$ as defined above consists of the SYT

123	124	134
4	3	2.

We now desire a symmetrised linear combination of the above set which is invariant under $Q\in S_1\otimes S_3\subset S_4$. Using elementary transpositions (2, 3) and (3, 4) and the simple form of the Young orthogonal representations for these, the right invariance criterion leads to the combination

$$\omega^{[3,1]}_{r\binom{122}{2}}=\frac{1}{3}\left\{\omega^{[3,1]}_{r\,\substack{123\\4}}+\sqrt{2}\omega^{[3,1]}_{r\,\substack{124\\3}}+\sqrt{6}\omega^{[3,1]}_{r\,\substack{134\\2}}\right\}. \tag{5}$$

We thus find that the symmetrisation coefficients are given by

$$a^{[3,1]}_{\substack{123\\4}\binom{122}{2}}=\frac{1}{3}\qquad a^{[3,1]}_{\substack{124\\3}\binom{122}{2}}=\frac{\sqrt{2}}{3}\qquad a^{[3,1]}_{\substack{134\\2}\binom{122}{2}}=\frac{\sqrt{2}}{\sqrt{3}}.$$

That the elementary transpositions of $S_{N_{11}}\otimes S_{N_{12}}\otimes\ldots\otimes S_{N_{nm}}$ are necessary and sufficient follows from the construction of $\omega^{\lambda}_{r(p)}$ as in the earlier work on $U(n)$ (Sarma and Sahasrabudhe 1980).

The right invariance of the operator combination $\omega^{\lambda}_{r(p)}$ of equation (3) as in equation (4) also ensures that we can relate any element of the linearly dependent set

$$\{\omega^{\lambda}_{rs}|(N_{11}\ldots N_{nm})\rangle;\ s\in S(p)\}\qquad\text{to}\qquad\omega^{\lambda}_{r(p)}|(N_{11}\ldots N_{nm})\rangle$$

through

$$\omega^{\lambda}_{rs}|(N_{11}\ldots N_{nm})\rangle=a^{\lambda}_{s(p)}\omega^{\lambda}_{r(p)}|(N_{11}\ldots N_{nm})\rangle. \tag{6}$$

We can readily verify this result using the example of equation (5). Since the monomial used is of the form $|\varphi_1\varphi_2^3\rangle$ which is invariant under the transpositions (2, 3) and (3, 4) we have, for example,

$$\omega_{r\,\substack{124\\3}}^{[3,1]}|\varphi_1\varphi_2^3\rangle = \omega_{r\,\substack{124\\3}}^{[3,1]}(3,4)|\varphi_1\varphi_2^3\rangle$$

$$= \left\{\tfrac{1}{3}\omega_{r\,\substack{124\\3}}^{[3,1]} + \frac{2\sqrt{2}}{3}\,\omega_{r\,\substack{123\\4}}^{[3,1]}\right\}|\varphi_1\varphi_2^3\rangle$$

leading to

$$\omega_{r\,\substack{123\\4}}^{[3,1]}|\varphi_1\varphi_2^3\rangle = \frac{1}{\sqrt{2}}\,\omega_{r\,\substack{124\\3}}^{[3,1]}|\varphi_1\varphi_2^3\rangle. \tag{7}$$

Similarly, using the transposition (2, 3) we also obtain

$$\omega_{r\,\substack{134\\2}}^{[3,1]}|\varphi_1\varphi_2^3\rangle = \sqrt{3}\omega_{r\,\substack{134\\2}}^{[3,1]}|\varphi_1\varphi_2^3\rangle. \tag{8}$$

Combining the results of equations (7) and (8) on the right of equation (5) we obtain the result

$$\omega_{r\,\substack{122\\2}}^{[3,1]}|\varphi_1\varphi_2^3\rangle = \frac{3}{\sqrt{2}}\omega_{r\,\substack{124\\3}}^{[3,1]}|\varphi_1\varphi_2^3\rangle.$$

This illustrates equation (6).

For a given value of the index r it can be shown (Sarma and Sahasrabudhe 1980, see also Patterson and Harter 1976b) that the set of functions

$$\left\{|[\lambda]; r(p)\rangle \equiv \left(\prod_{ij=11}^{nm} N_{ij}!\right)^{-1/2} \omega_{r(p)}^{\lambda}|(N_{11}\ldots N_{nm})\rangle\right\} \tag{9}$$

forms an orthonormal canonical basis set spanning the IRREP $[\lambda]$ of $U(nm)$, where the monomial $|(N_{11}\ldots N_{nm})\rangle$ of equation (1) corresponds to the Weyl tableau (p).

The transformations induced in these basis states by the generators $E_{ij,kt}$ ($ij, kt = 11, 12, \ldots, nm$) of $U(nm)$ can be obtained by defining them as shift operators in V_{nm} (Bohr and Mottelson 1969, p 121, Hecht 1973, Lezuo 1972, Patterson and Harter 1976b):

$$E_{ij,kt} = \sum_{\alpha=1}^{N} e_{ij,kt}(\alpha) \tag{10}$$

where the single particle operator $e_{ij,kt}(\alpha)$ annihilates the orbital χ_{kt} occupied by the αth particle and creates in its place the orbital χ_{ij}.

Based on the above considerations, we now consider the decomposition of the space V_{nm} as $V_n \otimes V_m$ where V_n and V_m are the fundamental representation spaces of $U(n)$ and $U(m)$ respectively. The existence of such a product decomposition of $U(nm)$ implies that each particle α is located by two coordinates r_α, θ_α. Accordingly each single particle orbital $\chi(\alpha)$ can be represented as an ordered product $u(r_\alpha)\,v(\theta_\alpha)$ where $u(r_\alpha)$ is a vector in V_n and $v(\theta_\alpha)$ is a vector in V_m. Thus if $\{u_i(r_\alpha)|i = 1, \ldots, n; \alpha = 1, \ldots, N\}$ and $\{v_j(\theta_\alpha)|j = 1, \ldots, m; \alpha = 1, \ldots, N\}$ define orthonormal bases spanning V_n and V_m respectively for the particles $\alpha = 1, 2, \ldots, N$, we have

$$\chi_{ij}(\alpha) = \chi_{ij}(r_\alpha, \theta_\alpha) = u_i(r_\alpha)v_j(\theta_\alpha). \tag{11}$$

Since every permutation of particles, P, now acts simultaneously on the r and θ

coordinates we have

$$P = P^r P^\theta \tag{12}$$

where P^r and P^θ are permutations of r and θ coordinates respectively. Using the above product form of the permutations, the Wigner operator defined by equation (2) becomes

$$\omega^\lambda_{rs} = \left(\frac{f_\lambda}{N!}\right)^{1/2} \sum_{P\in S_N} [P]^\lambda_{rs} P^r P^\theta. \tag{13}$$

Using the orthogonality of the Young representation matrices we also have (Kaplan 1975, p 42)

$$P^r = \sum_\mu \sum_{j_1,k_1=1}^{f_\mu} \left(\frac{f_\mu}{N!}\right)^{1/2} [P]^\mu_{j_1k_1} \omega^\mu_{j_1k_1} \tag{14}$$

$$P^\theta = \sum_\beta \sum_{j_2,k_2=1}^{f_\beta} \left(\frac{f_\beta}{N!}\right)^{1/2} [P]^\beta_{j_2k_2} \omega^\beta_{j_2k_2} \tag{15}$$

where $\omega^\mu_{j_1k_1}$ and $\omega^\beta_{j_2k_2}$ operate on the Nth rank tensor spaces $V_n \otimes^N$ and $V_m \otimes^N$ respectively. Substituting the right-hand sides of equations (14) and (15) on the right of equation (13) we obtain the results

$$\omega^\lambda_{rs} = \left(\frac{f_\lambda}{N!}\right)^{1/2} \sum_{\mu,\beta} \sum_{j_1,k_1}^{f_\mu} \sum_{j_2,k_2}^{f_\beta} \left(\frac{f_\mu}{N!}\right)^{1/2} \left(\frac{f_\beta}{N!}\right)^{1/2} \left(\sum_{P\in S_N} [P]^\lambda_{rs}[P]^\mu_{j_1k_1}[P]^\beta_{j_2k_2}\right) \omega^\mu_{j_1k_1}\omega^\beta_{j_2k_2}. \tag{16}$$

Using the definition of the Clebsch–Gordan (CG) coefficients for S_N (Hamermesh 1962, p 261) we have

$$\sum_{P\in S_N} [P]^\lambda_{rs}[P]^\mu_{j_1k_1}[P]^\beta_{j_2k_2} = \left(\frac{N!}{f_\lambda}\right) \sum_{\tau_\lambda} \begin{pmatrix} \lambda\tau_\lambda & \mu & \beta \\ r & j_1 & j_2 \end{pmatrix} \begin{pmatrix} \lambda\tau_\lambda & \mu & \beta \\ s & k_1 & k_2 \end{pmatrix} \tag{17}$$

where τ_λ is an index to distinguish between the multiple occurrence of the IRREP $[\lambda]$ of S_N in the product representations $[\mu]\times[\beta]$ of $S_N\times S_N$. Using the result of equations (17) in (16) we obtain the result

$$\omega^\lambda_{rs} = (N!f_\lambda)^{-1/2} \sum_{\mu,\beta} \sum_{\tau_\lambda} \sum_{j_1,k_1}^{f_\mu} \sum_{j_2,k_2}^{f_\beta} (f_\mu f_\beta)^{1/2} \begin{pmatrix} \lambda\tau_\lambda & \mu & \beta \\ r & j_1 & j_2 \end{pmatrix} \begin{pmatrix} \lambda\tau_\lambda & \mu & \beta \\ s & k_1 & k_2 \end{pmatrix} \omega^\mu_{j_1k_1}\omega^\beta_{j_2k_2}. \tag{18}$$

The right-hand side of equation (18) is completely determinable since reasonably efficient schemes are available for determining the Clebsch–Gordan coefficients of S_N (Schindler and Mirman 1977, Sahasrabudhe *et al* 1981). As an illustration of the form the right-hand side of equation (18) takes, we consider a particular Wigner operator for the IRREP [2, 1] of S_3 and express it as a linear combination of product representations occurring in $S_3\times S_3$

$$\begin{aligned} \omega^{[2,1]}_{\substack{13\;12\\2\;\;\;3}} = \frac{1}{\sqrt{12}} \Big\{ & \sqrt{2}\,\omega^{[3]}_{123\;123} \times \omega^{[2,1]}_{\substack{13\;12\\2\;\;\;3}} + \sqrt{2}\,\omega^{[2,1]}_{\substack{13\;12\\2\;\;\;3}} \times \omega^{[3]}_{123\;123} \\ & + \omega^{[2,1]}_{\substack{12\;13\\3\;\;\;2}} \times \omega^{[2,1]}_{\substack{13\;13\\2\;\;\;2}} + \omega^{[2,1]}_{\substack{13\;13\\2\;\;\;2}} \times \omega^{[2,1]}_{\substack{12\;13\\3\;\;\;2}} - \omega^{[2,1]}_{\substack{12\;12\\3\;\;\;3}} \times \omega^{[2,1]}_{\substack{13\;12\\2\;\;\;3}} \\ & - \omega^{[2,1]}_{\substack{13\;12\\2\;\;\;3}} \times \omega^{[2,1]}_{\substack{12\;12\\3\;\;\;3}} + \sqrt{2}\,\omega^{[1^3]}_{\substack{1\,1\\2\,2\\3\,3}} \times \omega^{[2,1]}_{\substack{12\;13\\3\;\;\;2}} + \sqrt{2}\,\omega^{[2,1]}_{\substack{12\;13\\3\;\;\;2}} \times \omega^{[1^3]}_{\substack{1\,1\\2\,2\\3\,3}} \Big\} \end{aligned}$$

where the CG coefficients listed by Hamermesh (1962) have been used.

Applying the operators on both sides of equation (18) to the tensor monomial of equation (1) and using equation (6) we obtain

$$\omega^{\lambda}_{r(p)}|(N_{11}\ldots N_{nm})\rangle=(N!f_\lambda)^{-1/2}(a^{\lambda}_{s(p)})^{-1}\sum_{\mu,\beta}\sum_{\tau_\lambda}\sum_{j_1,k_1}^{f_\mu}\sum_{j_2,k_2}^{f_\beta}(f_\mu f_\beta)^{1/2}$$
$$\times\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ r & j_1 & j_2\end{pmatrix}\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ s & k_1 & k_2\end{pmatrix}\omega^{\mu}_{j_1k_1}\omega^{\beta}_{j_2k_2}|(N_{11}\ldots N_{nm})\rangle. \qquad (19)$$

We now examine the right-hand side of equation (19) in greater detail. Each element of the set defined by equation (1) spanning $V_{nm}\otimes^N$ can be decomposed into a product of an Nth rank tensor in $V_n\otimes^N$ and an Nth rank tensor in $V_m\otimes^N$. In view of the ordering used for defining the elements of equation (1), we find that the tensor of $V_n\otimes^N$ is already in proper order with u_i preceding u_j if $i<j$. This, however, is not in general true for the element of $V_m\otimes^N$ occurring in $V_{nm}\otimes^N$. As an example consider the third rank tensor $|(\chi_{11}\chi_{22}\chi_{31})\rangle$ generated from products in $V_9=V_3\otimes V_3$. We have

$$|(\chi_{11}\chi_{22}\chi_{31})\rangle=|(u_1u_2u_3)\rangle|(v_1v_2v_1)\rangle.$$

The right-hand side of the above expression which is not properly ordered can be restored to proper ordering using a coordinate permutation in θ space as

$$|(\chi_{11}\chi_{22}\chi_{31})\rangle=|(u_1u_2u_3)\rangle(2,3)^{-1}|(v_1^2v_2)\rangle.$$

Generalising the above result we have

$$|(N_{11}\ldots N_{nm})\rangle\equiv|(N_{(ij)})\rangle=|(N^u_1\ldots N^u_n)\rangle P^{-1}|(N^v_1\ldots N^v_m)\rangle\equiv|(N^u_{(i)})\rangle P^{-1}|(N^v_{(j)})\rangle \qquad (20)$$

where P is the permutation in θ space which properly orders the tensor of $V_m\otimes^N$. We now note that the right application of any permutation to a Wigner operator leads to the result

$$\omega^{\beta}_{j_2k_2}P^{-1}=\left(\frac{f_\beta}{N!}\right)^{1/2}\sum_Q[Q]^{\beta}_{j_2k_2}QP^{-1}=\sum_{k_2'}[P]^{\beta}_{k_2'k_2}\omega^{\beta}_{j_2k_2'}. \qquad (21)$$

Combining the results of equations (20) and (21) on the right-hand side of equation (19) we obtain

$$|[\lambda];r(s:N_{11}\ldots N_{nm})\rangle=\left(N!f_\lambda\prod_{ij=11}^{nm}N_{ij}!\right)^{-1/2}(a^{\lambda}_{s(s:N_{(ij)})})^{-1}$$
$$\times\sum_{\mu,\beta}\sum_{\tau_\lambda}\sum_{j_1,k_1}^{f_\mu}\sum_{j_2,k_2,k_2'}^{f_\beta}(f_\mu f_\beta)^{1/2}[P]^{\beta}_{k_2'k_2}\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ r & j_1 & j_2\end{pmatrix}\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ s & k_1 & k_2\end{pmatrix}$$
$$\times\omega^{\mu}_{j_1k_1}|(N^u_1\ldots N^u_n)\rangle\omega^{\beta}_{j_2k_2'}|(N^v_1\ldots N^v_m)\rangle \qquad (22)$$

where we have replaced the Weyl index (p) by $(s:N_{(ij)})$ on the left as outlined earlier.

Symmetrising the Wigner operators occurring on the right-hand side of equation (22) as was done using equation (6) for the basis states of U(nm) we have

$$|[\lambda];r(s:N_{(ij)})\rangle=\mathcal{N}\sum_{\mu,\beta}\sum_{\tau_\lambda}\sum_{j_1,j_2}^{f_\mu}\sum_{j_2,k_2,k_2'}^{f_\beta}(f_\mu f_\beta)^{1/2}[P]^{\beta}_{k_2'k_2}$$
$$\times\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ r & j_1 & j_2\end{pmatrix}\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ s & k_1 & k_2\end{pmatrix}a^{\mu}_{k_1(k_1:N^u_{(i)})}a^{\beta}_{k_2'(k_2':N^v_{(j)})}$$
$$\times|[\mu];j_1(k_1:N^u_{(i)})\rangle|[\beta];j_2(k_2':N^v_{(j)})\rangle \qquad (23)$$

where

$$\mathcal{N}=(a^{\lambda}_{s(s:N_{(ij)})})^{-1}\left(\frac{\Pi_i\,(N^u_i!)\,\Pi_j(N^v_j!)}{\Pi_{ij}\,(N_{ij}!)N!f_\lambda}\right)^{1/2} \tag{24}$$

and where, further, we have abbreviated all the relevant monomial tensors as

$$|N_{(ij)}\rangle\equiv|(N_{11}\ldots N_{nm})\rangle=|(N^u_1\ldots N^u_n)\rangle P^{-1}|(N^v_1\ldots N^v_m)\rangle$$
$$\equiv|N^u_{(i)}\rangle P^{-1}|N^v_{(j)}\rangle.$$

For notational convenience we may drop the repeated index s specifying the SYT on the symmetrisation coefficients wherever no confusion is likely to arise, as for example, in the replacement of $a^{\lambda}_{s(s:N_{(ij)})}$ by $a^{\lambda}_{s(N_{(ij)})}$.

At first sight it might appear surprising that in equation (23) we are using canonical tensor basis states carrying two indices j_1, k_1 etc. That such states occur naturally in any scheme utilising the duality between unitary and permutation groups has been pointed out by a number of workers (Kaplan 1975, Bohr and Mottelson 1969). In fact Bohr and Mottelson display some of these states explicitly (cf p 130). As pointed out by them, and demonstrated explicitly later (Sarma and Sahasrabudhe 1980), the generators of the unitary group change only the monomial and hence affect the indices k_1, k_2' leaving j_1 and j_2 unaffected. This result combined with the definition of the CG coefficients of S_N (cf Kaplan 1975, equation (1.79)) when used to perform the summation over j_1 and j_2 on the right of equation (23) leads to

$$|[\lambda];r(s:N_{(ij)})\rangle=\mathcal{N}\sum_{\mu,\beta}\sum_{\tau_\lambda}\sum_{k_1}^{f_\mu}\sum_{k_2,k_2'}^{f_\beta}(f_\mu f_\beta)^{1/2}[P]^{\beta}_{k_2'k_2}\begin{pmatrix}\lambda\tau_\lambda & \mu & \beta\\ s & k_1 & k_2\end{pmatrix}a^{\mu}_{k_1(N^u_{(i)})}a^{\beta}_{k_2'(N^v_{(j)})}$$
$$\times|[\mu];\begin{Bmatrix}\lambda\tau_\lambda\\ r\end{Bmatrix}(k_1:N^u_{(i)})\rangle|[\beta];\begin{Bmatrix}\lambda\tau_\lambda\\ r\end{Bmatrix}(k_2':N^v_{(j)})\rangle \tag{25}$$

where now the product states are in the usual form with fixed first indices $\{^{\lambda\tau_\lambda}_{r}\}$ and, in general, more than one value of $k_1(k_2')$ can produce the same state

$$|[\mu];\begin{Bmatrix}\lambda\tau_\lambda\\ r\end{Bmatrix}(k_1:N^u_{(i)})\rangle\Big(|[\beta];\begin{Bmatrix}\lambda\tau_\lambda\\ r\end{Bmatrix}(k_2':N^v_{(j)})\rangle\Big).$$

The right-hand side of equation (25) seems to imply that the linear combinations of product states depend on the SYT index s from which we start. This is, in fact, not true. Replacement of the symmetrised operator on the left by $(a^{\lambda}_{s'(p)})^{-1}\omega^{\lambda}_{rs'}$, for $s'\neq s$ in accordance with equation (6) and expansion of $\omega^{\lambda}_{rs'}$ using equation (18) shows immediately that such dependence on starting SYT does not exist. This permits us to identify each of the states on the left-hand side of equation (25) with the standard canonical tensor basis which, in turn, can be identified with the corresponding Weyl tableau basis. For computational purposes it is convenient to replace the summations on the right of equation (25) with equivalent ones as follows. We replace $\Sigma^{f_\mu}_{k_1}$ by $\Sigma_{(p^u)\in(p)}\,\Sigma_{k_1\in S(p^u)}$ where the first summation is over all Weyl tableaux (p^u) associated with a given monomial $|N^u_{(i)}\rangle$ contained in the $|N_{(ij)}\rangle$ leading to a specified (p). A similar replacement is done for the summation over k_2'. Such replacements are possible because the SYT $k_1(k_2')$ are classified as disjoint sets $S(p^u)$ $(S(p^v))$ corresponding to distinct $(p^u)((p^v))$ associated with a given monomial $|N^u_{(i)}\rangle(|N^v_{(j)}\rangle)$. Thus we have

$$|[\lambda]; r(p)\rangle = \mathcal{N} \sum_{\mu,\beta} \sum_{\tau_\lambda} (f_\mu f_\beta)^{1/2} \sum_{(p^u)\in(p)} \sum_{(p^v)\in(p)} \times \left\{ \sum_{k_1\in S(p^u)} \sum_{k_2'\in S(p^v)} \sum_{k_2=1}^{f_\beta} \begin{pmatrix}\lambda\tau_\lambda & \mu & \beta \\ s & k_1 & k_2\end{pmatrix} [P]^{\beta}_{k_2'k_2} a^{\mu}_{k_1(p^u)} a^{\beta}_{k_2'(p^v)} \right\} \times |[\mu]; \begin{Bmatrix}\lambda\tau_\lambda \\ r\end{Bmatrix}(p^u)\rangle |[\beta]; \begin{Bmatrix}\lambda\tau_\lambda \\ r\end{Bmatrix}(p^v)\rangle \tag{26}$$

where s is any SYT in $S(p)$ and the monomials $|N_{(ij)}\rangle$, $|N^u_{(i)}\rangle$ and $|N^v_{(j)}\rangle$, corresponding to (p), (p^u) and (p^v) respectively, are related by $|N_{(ij)}\rangle = |N^u_{(i)}\rangle P^{-1} |N^v_{(j)}\rangle$.

We now illustrate the use of equation (26) by considering a simple example of the basis state $\begin{smallmatrix}11 & 22\\ 22 & \end{smallmatrix}$ of U(4) which will be expressed in terms of the basis states spanning the product representations of the subgroup U(2)⊗U(2). The corresponding IRREP [2, 1] of S_3 occurs in the reduction of the inner product IRREPS [3]×[2, 1], [2, 1]×[3], [2, 1]×[2, 1], $[1^3]$×[2, 1], [2, 1]×$[1^3]$ of $S_3 \times S_3$. Out of these the last two do not contribute to the relevant basis state of U(4) since the Wigner operator for the IRREP $[1^3]$ annihilates the tensor monomials $|u_1\, u_2^2\rangle$ and $|v_1\, v_2^2\rangle$ occurring in $|\chi_{11}\chi_{22}^2\rangle$. We further observe that the matching permutation P in v space is the identity since the required monomial is already properly ordered. Taking $\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}$ as the reference tableau s on the left of equation (25), the necessary CG coefficients (cf Hamermesh 1962, p 270) are

$$\begin{pmatrix}[2,1] & [3] & [2,1] \\ \begin{smallmatrix}1&2\\3&\end{smallmatrix} & 1\;2\;3 & \begin{smallmatrix}1&2\\3&\end{smallmatrix}\end{pmatrix} = \begin{pmatrix}[2,1] & [2,1] & [3] \\ \begin{smallmatrix}1&2\\3&\end{smallmatrix} & \begin{smallmatrix}1&2\\3&\end{smallmatrix} & 1\;2\;3\end{pmatrix} = 1$$

$$\begin{pmatrix}[2,1] & [2,1] & [2,1] \\ \begin{smallmatrix}1&2\\3&\end{smallmatrix} & \begin{smallmatrix}1&2\\3&\end{smallmatrix} & \begin{smallmatrix}1&2\\3&\end{smallmatrix}\end{pmatrix} = -\begin{pmatrix}[2,1] & [2,1] & [2,1] \\ \begin{smallmatrix}1&2\\3&\end{smallmatrix} & \begin{smallmatrix}1&3\\2&\end{smallmatrix} & \begin{smallmatrix}1&3\\2&\end{smallmatrix}\end{pmatrix} = \frac{1}{\sqrt{2}}.$$

Noting that for the given monomial we have $N_{11} = N_1^u = N_1^v = 1$ and $N_{22} = N_2^u = N_2^v = 2$ with all others being zero, we have, on using equation (26),

$$\left|\begin{array}{|c|c|}\hline 11 & 22 \\ \hline 22 & \\ \cline{1-1}\end{array}\right\rangle = \left(\frac{1}{2}\right)^{-1} \left(\frac{1!2!1!2!}{1!2!3!2}\right)^{1/2} \left\{ \sqrt{2}.1.1.\frac{1}{2} \left|\begin{array}{|c|c|c|}\hline u_1 & u_2 & u_2 \\ \hline\end{array}\right\rangle \left|\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_2 & \\ \cline{1-1}\end{array}\right\rangle \right.$$

$$+\sqrt{2}.1.\tfrac{1}{2}.1 \left|\begin{array}{|c|c|}\hline u_1 & u_2 \\ \hline u_2 & \\ \cline{1-1}\end{array}\right\rangle \left|\begin{array}{|c|c|c|}\hline v_1 & v_2 & v_2 \\ \hline\end{array}\right\rangle + 2\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right)$$

$$\left.\times \left|\begin{array}{|c|c|}\hline u_1 & u_2 \\ \hline u_2 & \\ \cline{1-1}\end{array}\right\rangle \left|\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_2 & \\ \cline{1-1}\end{array}\right\rangle \right\}$$

$$= \frac{1}{\sqrt{3}} \left\{ \left|\begin{array}{|c|c|c|}\hline u_1 & u_2 & u_2 \\ \hline\end{array}\right\rangle \left|\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_2 & \\ \cline{1-1}\end{array}\right\rangle + \left|\begin{array}{|c|c|}\hline u_1 & u_2 \\ \hline u_2 & \\ \cline{1-1}\end{array}\right\rangle \left|\begin{array}{|c|c|c|}\hline v_1 & v_2 & v_2 \\ \hline\end{array}\right\rangle \right.$$

$$\left. - \left|\begin{array}{|c|c|}\hline u_1 & u_2 \\ \hline u_2 & \\ \cline{1-1}\end{array}\right\rangle \left|\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_2 & \\ \cline{1-1}\end{array}\right\rangle \right\}$$

where the Weyl tableau notation has been used and the symmetrisation coefficients

$$a^{[2,1]}_{\substack{1\,2\\3}\left(\substack{11\;22\\22}\right)}=\frac{1}{2} \qquad a^{[3]}_{123\,(122)}=1 \qquad a^{[2,1]}_{\substack{12\\3}\left(\substack{12\\2}\right)}=\frac{1}{2} \qquad a^{[2,1]}_{\substack{13\\2}\left(\substack{12\\2}\right)}=\frac{\sqrt{3}}{2} \tag{27}$$

have been obtained using the procedure outlined earlier.

Two special cases of equation (25) which have important bearing on many physical applications are the study of the identity and alternating representations of U(nm). In both these cases the required CG coefficients of S_N have a particularly simple form and this, in turn, leads to a particularly simple form for equation (25). We consider first the identity representation $[N]$ of U(nm). The CG coefficients for the corresponding IRREP $[N]$ of S_N are of the form (cf Hamermesh 1962, equation (7–209))

$$\begin{pmatrix} [N] & \mu & \beta \\ 1 & k_1 & k_2 \end{pmatrix} = \frac{1}{\sqrt{f_\mu}}\delta_{\mu\beta}\,\delta_{k_1k_2}. \tag{28}$$

Using this in equation (25) we obtain the result

$$|[N]; 1(1:N_{11}\dots N_{nm})\rangle = \mathcal{N}\sum_{\mu}\sum_{k_1,k_2}^{f_\mu}(f_\mu)^{1/2}[P]^{\mu}_{k_2k_1}a^{\mu}_{k_1(N^u_{(i)})}$$
$$\times a^{\mu}_{k_2(N^v_{(j)})}|[\mu];\begin{Bmatrix}[N]\\1\end{Bmatrix}(k_1:N^u_1\dots N^u_n)\rangle|[\mu];\begin{Bmatrix}[N]\\1\end{Bmatrix}(k_2:N^v_1\dots N^v_m)\rangle \tag{29}$$

where $\mathcal{N}$ is as defined by equation (24) with $f_\lambda = f_{[N]} = 1$. This particular representation has been extensively studied using a boson operator approach (cf Patterson and Harter 1976b and references therein). However, it is to be noted that this approach does not lead to closed expressions as above but involves operations on specific basis functions of U(nm). As an illustration of equation (29) we consider the case $N = 2$, $n = 3$, $m = 2$ and the representation [3] of U(6). Let the monomial of $V_6\otimes^3$ being considered be $|(\chi_{11}\chi_{22}\chi_{32})\rangle = |(u_1\ u_2\ u_3)\rangle|(v_1\ v_2\ v_2)\rangle$. We note that there is no symmetrisation required in the product space basis $|(\chi_{11}\chi_{22}\chi_{32})\rangle$ since each of the χ is singly occupied. Thus $a^{[3]}_{123(11\,22\,32)} = 1$. The same result holds true in the u space since each orbital defining the monomial $|(u_1\ u_2\ u_3)\rangle$ is again singly occupied. Thus each of the required symmetrisation coefficients $a^{\mu}_{k_1(N^u_{(i)})} = 1$ for this monomial. The occupancies of the orbitals defining the v-space monomial are the same as those considered in equation (19), so that the required symmetrisation coefficients are the last three given in that equation. Using these results and $f_{[3]} = 1$, $f_{[2,1]} = 2$ on the right-hand side of equation (29) we obtain

$$|[3]; 1(1{:}\ 11\ 22\ 32)\rangle = \frac{1}{\sqrt{3}}\Big\{|[3]; (1{:}\ u_1u_2u_3)\rangle|[3]; (1{:}\ v_1v_2^2)\rangle$$
$$+\frac{1}{\sqrt{2}}|[2,1]; 1(1{:}\ u_1u_2u_3)\rangle|[2,1]; 1(1{:}\ v_1v_2^2)\rangle$$
$$+\frac{\sqrt{3}}{\sqrt{2}}|[2,1]; 1(2{:}\ u_1u_2u_3)\rangle|[2,1]; 1(1{:}\ v_1v_2^2)\rangle\Big\}.$$

A similar procedure can be used to obtain the other two basis states listed by Patterson and Harter (1976b) using the appropriate reordering permutations in the v space. This procedure is found to lead to the inverse of the real orthogonal matrix given by them (cf their equation (24)). A similar check is possible with the results given by Lichtenberg (1970) (cf equation (11.82)).

As a final step in the present study we obtain the basis for the alternating representation $[1^N]$ of U(nm) from those spanning the product representations of U(n)⊗U(m). In this case we again start with equation (25) and use the CG coefficients for S_N (cf Hamermesh 1962 equation (7–211a)):

$$\begin{pmatrix} [1^N] & \mu & \bar{\mu} \\ 1 & k_1 & k_2 \end{pmatrix} = \frac{1}{\sqrt{f_\mu}} \Lambda^{\mu}_{k_1} \delta_{\bar{k}_1 k_2} \tag{30}$$

where the SYT k_2 is obtained from k_1 by interchanging the rows and the columns and $\Lambda^{\mu}_{k_1} = \pm 1$ depending on whether k_1 has been obtained from the first of the SYT spanning the given IRREP by an even or an odd permutation. Using the results of equation (30) on the right-hand side of equation (25) we obtain

$$|[1^N]; 1(1: N_{11} \ldots N_{nm})\rangle = \mathcal{N} \sum_{\mu} \sum_{k_1,k_2} (f_\mu)^{1/2} [P]^{\bar{\mu}}_{k_2 \bar{k}_1} \Lambda^{\bar{\mu}}_{k_2} a^{\bar{\mu}}_{k_2(N^u_{(j)})}$$
$$\times a^{\mu}_{k_1(N^u_{(i)})} |[\mu]; \begin{Bmatrix} [1^N] \\ 1 \end{Bmatrix} (k_1: N^u_1 \ldots N^u_n)\rangle |[\bar{\mu}]; \begin{Bmatrix} [1^N] \\ 1 \end{Bmatrix} (k_2: N^v_1 \ldots N^v_m)\rangle. \tag{31}$$

This result may be used to reproduce equation (48) of Patterson and Harter (1976b).

3. Discussion

To the best of our knowledge equations (25) and (26) constitute the first attempt to obtain a generalised expression for obtaining the basis transformation for the subgroup adaptation U(nm)↓U(n)⊗U(m). This expression, valid for any n, m and N and any IRREP $[\lambda]$ of U(nm), could be derived mainly because the $S_N \times S_N$ content of U(n)⊗U(m) could be separated out. In the usual Gelfand–Zetlin (1950) form for the canonical basis spanning the IRREPS of unitary groups this separation is not readily evident. In fact even the explicit realisation of more than one set of canonical basis for these IRREPS of unitary groups appears only if we employ the tensor representation (cf Bohr and Mottelson 1969, p 130). A computation scheme based on equations (25) and (26) is quite feasible since a number of useful algorithms have been obtained recently which enable permutation groups to be handled readily (Sarma and Rettrup 1977, Rettrup 1977, Sahasrabudhe *et al* 1980, 1981, Schindler and Mirman 1977, Karwowski 1973, Sarma 1981, Butler 1975). Further the specialised forms given in equations (29) and (31) which involve only a knowledge of symmetrisation coefficients and the representation matrix of an appropriate permutation are the most useful ones in many areas of application.

Attempts are at present being made to determine the matrix elements of irreducible tensor operators of U(nm) using these basis states adapted to U(n)⊗U(m).

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